Exercise 40

Find the work done by the force field $\mathbf{F}(x, y) = x^2\,\mathbf{i} + ye^x\,\mathbf{j}$ on a particle that moves along the parabola $x = y^2 + 1$ from $(1, 0)$ to $(2, 1)$.

Solution

Parameterize the path that the particle moves on by $x = t^2 + 1$, $y = t$, $z = 0$ so that

$$\mathbf{r}(t) = \langle t^2 + 1, t, 0\rangle, \quad 0 \le t \le 1.$$

Calculate the line integral of the force field over the parabolic path to find the work done.

$$\begin{aligned}
W &= \int_C \mathbf{F}\cdot d\mathbf{r} \\
&= \int_0^1 \mathbf{F}(\mathbf{r}(t))\cdot \mathbf{r}'(t)\,dt \\
&= \int_0^1 \left\langle (t^2+1)^2, te^{t^2+1}, 0\right\rangle \cdot \langle 2t, 1, 0\rangle\,dt \\
&= \int_0^1 [(t^2+1)^2(2t) + te^{t^2+1}(1) + (0)(0)]\,dt \\
&= \int_0^1 [2t(t^2+1)^2 + te^{t^2+1}]\,dt \\
&= 2\int_0^1 t(t^2+1)^2\,dt + \int_0^1 te^{t^2+1}\,dt
\end{aligned}$$

Make the following substitution in both integrals.

$$u = t^2 + 1$$

$$du = 2t\,dt \quad \to \quad \frac{du}{2} = t\,dt$$

Consequently,

$$\begin{aligned}
W &= 2\int_{0^2+1}^{1^2+1} u^2\left(\frac{du}{2}\right) + \int_{0^2+1}^{1^2+1} e^u\left(\frac{du}{2}\right) \\
&= \int_1^2 u^2\,du + \frac{1}{2}\int_1^2 e^u\,du \\
&= \left(\frac{u^3}{3}\right)\Big|_1^2 + \frac{1}{2}(e^u)\Big|_1^2 \\
&= \left(\frac{2^3}{3} - \frac{1^3}{3}\right) + \frac{1}{2}(e^2 - e^1) \\
&= \frac{7}{3} + \frac{e}{2}(e-1).
\end{aligned}$$